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XXXVII.—ON THE IMAGINARY ROOTS OF NUMERICAL EQUATIONS, WITH AN INVESTIGATION AND PROOF OF NEWTON'S RULE. By J. R. YOUNG, Esq., formerly Professor of Mathematics in Belfast College.

[Read November 9, 1868.]

(1) LET the general equation of the nth degree, with numerical coefficients, be represented by

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots + A_2 x^2 + A_1 x + A_0 = 0 \quad \ldots \; [I],$$

and let it be transformed into another by substituting $x + r$ for x. Then if r be determined by the condition that the second coefficient in the transformed equation shall be zero, the third coefficient will be found to be

$$\frac{A_{n-2}}{A_n} - \frac{n(n-1)}{2}\left(\frac{A_{n-1}}{nA_n}\right)^2;$$

consequently, if the sign of this third coefficient be the same as that of the first, the first three terms of the transformed equation,—the middle term being zero, will satisfy the condition of De Gua,* and will therefore imply the existence of at least one pair of imaginary roots in the transformed, and therefore of one pair also in the original equation. Hence, multiplying this third coefficient by the positive quantity $2nA_n^2$, a pair of imaginary roots will be indicated, provided the first three coefficients of the proposed equation satisfy the condition

$$2nA_{n-2}A_n > (n-1)A^2_{n-1} \quad \ldots \; [1].$$

(2) If the order of the coefficients in [I] be reversed, we shall have an equation the roots of which will be the reciprocals of the roots of [I]: the existence of a pair of imaginary roots in either equation implies therefore the existence of a pair, the reciprocals of those roots, in the other. Consequently the criterion [1] may be applied, as a test, as well to the last three coefficients of [I] as to the first three; so that a pair of imaginary roots in [I] will equally be indicated, provided that the condition

$$2nA_0A_2 > (n-1)A_1^2 \quad \ldots \; [2]$$

be satisfied.

(3) Now, it is well known that if a limiting or derived equation have imaginary roots, the primitive equation must also have imaginary roots,—as many at least: taking, therefore, the several limiting equations, derived one after another, each from the immediately preceding, in the usual way, [I] being the primitive, and applying the criterion

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* Conformably to usage, I have called this "the condition of De Gua;" but it is implied in the Rule of Newton, published many years before the researches of De Gua appeared.

[2] to the last three coefficients of each, we shall arrive at the series of conditions which here follow; and the existence of any one of these conditions will imply the existence of at least one pair of imaginary roots in the primitive equation.

Conditions of imaginary roots in [I].

$$\begin{aligned} 2n\mathrm{A}_0A_2 &> (n-1)A_1^2 \\ 3(n-1)A_1A_3 &> 2(n-2)A_2^2 \\ 4(n-2)A_2A_4 &> 3(n-3)A_3^2 \quad \ldots [3] \\ 5(n-3)A_3A_5 &> 4(n-4)A_4^2. \\ &\vdots \\ 2nA_{n-2}A_n &> (n-1)A^2_{n-1}. \end{aligned}$$

(4) From a mere inspection of this group of conditions, it is obvious that all are comprehended in the general formula

$$(m+1)\ (n-\overline{m-1})\ A_{m-1}A_{m+1} > m(n-m)\ A^2_m \ldots . [4],$$

where m is the exponent of x in the middle one of any three consecutive terms of the equation [I]. And from this general formula we at once see that the difference of the numerical multipliers

$$(m+1)\ (n-\overline{m-1}), \text{ and } m(n-m)$$

is always the same for the same value of n, namely, $n+1$. We may therefore express the above conditions somewhat differently, thus:—

$$\begin{aligned} [(n-1)+(n+1)]\ A_0A_2 &> (n-1)\ A_1^2 \\ [2(n-2)+(n+1)]\ A_1A_3 &> 2(n-2)\ A_2^2 \\ [3(n-3)+(n+1)]\ A_2A_4 &> 3(n-3)\ A_3^2 . \ . \ . [5] \\ [4(n-4)+(n+1)]\ A_3A_5 &> 4(n-4)\ A_4^2 \\ &\vdots \\ [(n-1)+(n+1)]\ A_{n-2}A_n &> (n-1)\ A_n{}^2{}_{-1}; \end{aligned}$$

so that the multiplier for the square of the middle term of any triad of terms, increased by the constant number $n+1$, will always be the multiplier for the product of the extreme terms; and therefore, in applying the tests, it will usually be the more convenient to deal with the squares first.

(5) It is evident that the foregoing inequalities, for an equation of the nth degree, are $n-1$ in number; and from the general expression [4] we see that the multiplier which enters the middle term of each of the two completed series of terms, when n is even, or the number of terms in each completed series odd, is always a square number: thus,

putting $2m$ for n, the middle term will be the mth, and therefore, by the general form referred to, the proper multipliers will be

$$(m+1)\,(2m-\overline{m-1}) = (m+1)^2, \text{ and } m(2m-m) = m^2.$$

Of course, since the multipliers in each completed series taken in order from first to last are the same as when taken in order from last to first, the product of the extremes, as also of any two multipliers equidistant from the extremes, will be a square number.

(6) If in any one of the foregoing conditions, the first member should be equal to, instead of greater than, the second, a pair of imaginary roots will still be implied. For in that case, not only does the second coefficient vanish in the equation in which the condition of equality has place, but the third coefficient also—the general expression above, for the third coefficient, being then zero; and we know that consecutive zeros always imply imaginary roots.

(7) Any one of the group of conditions [3], involving three consecutive coefficients of the primitive equation [*I*], being taken, if that condition hold or fail, it will, in like manner, hold or fail for the terms involving the same three coefficients in every limiting equation derived from the primitive. This is proved as follows:—

The general expression for any triad of consecutive terms in [*I*] is

$$A_{m+1}x^{m+1} + A_m x^m + A_{m-1}x^{m-1}.$$

The triad derived from this is

$$(m+1)\,A_{m+1}x^m + m\,A_m x^{m-1} + (m-1)\,A_{m-1}x^{m-2},$$

and the property affirmed is—that according as the condition in [4] holds or fails for the above primitive triad, so will the following condition, in which the derived triad replaces the former, hold or fail, namely, the condition

$$m(n-\overline{m-2})\,(m-1)\,(m+1)\,A_{m-1}\,A_{m+1} > (m-1)\,(n-\overline{m-1})\,(mA_m)^2,$$

or, expunging the factors common to both sides, the condition

$$(n-\overline{m-2})\,(m+1)\,A_{m-1}\,A_{m+1} > m(n-\overline{m-1})A^2_m.$$

For, it being remembered that, for the triad with which we are now dealing, the degree of the equation is $n-1$, and not n (as in the case of the primitive triad), we must put $n-1$ for n in this expression: its form will then be

$$(n-1-\overline{m-2})\,(m+1)\,A_{m-1}A_{m+1} > m(n-1-\overline{m-1})\,A^2_m.$$

which is the same as

$$(m+1)\,(n-\overline{m-1})\,A_{m-1}A_{m+1} > m(n-m)\,A^2_m,$$

that is, *it is identical with the form* [4]. And, as a corollary to this theorem, it follows that when the two expressions [3] are equal instead of unequal for any triad in a derived equation, they must be equal for the corrseponding triad in the preceding equation, and conversely; and from this it further follows that in the development of $(x+a)^n$, all the triads must satisfy the conditions of equality. For they are all satisfied in the case

$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3,$$

and consequently, from the above inference, all (except the last triad, which has no correspondent here) must satisfy the conditions of equality in

$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4,$$

of which the former multiplied by 4 is the derived equation; and thence the conditions of equality are satisfied in $(x+a)^5$, $(x+a)^6$, &c., by whatever factors these be multiplied. And since in each case the last triad always satisfies the same condition as the first, it follows that *all* the triads in the development of $A_n(x+a)^n$ satisfy the conditions of equality; and this development being fixed in form, these tests of equality may be employed to ascertain whether a polynomial is really the development of a binomial, or of a binomial multiplied by a factor, or not.

(8) From the theorem just established, we see that whatever imaginary roots the conditions [3] may enable us to detect the existence of in the equation [*I*], these roots always have a peculiar character; they are distinguished from other imaginary roots in this,—namely, that on account of their entrance into the primitive equation, imaginarity* is *necessarily* transmitted to the first derived equation, thence to the second derived equation, and so on, till one of the three coefficients of the primitive, which supply the condition [3], disappears, as it at length must do in the process of successive derivation. As long as the three coefficients fulfilling one of the conditions [3] are all preserved in the subsequent equations, so long will each of those equations have a pair of imaginary roots. With regard, therefore, to this particular class of imaginary roots, it is true not only that a pair in any derived equation implies, of necessity, the entrance of a pair in the primitive, but, conversely, that a pair in the primitive necessitates a pair in every derived equation down to that one from which the third coefficient in the primitive triad has disappeared.

With other imaginary pairs, that is, with those pairs the existence of which is not indicated by any of the conditions [3], the case is different: we know that the equation may have imaginary roots, and yet not transmit imaginarity to any of the equations derived from it—the roots of these may all be real; if a pair of them be imaginary, then, indeed, a pair also imaginary must necessarily enter the primitive equation; but the converse of this is not true, except under the peculiar circumstances noticed above; that is, except when the original coëfficients satisfy one or more of the conditions [3]. It is exclusively with this class of imaginary roots that we propose at present to deal: we shall have nothing to do—at least till notice is given—with any of the imaginary roots of an equation, the existence of which roots is not discoverable from the mere coefficients of that equation when submitted to the tests of imaginarity marked [3] or [5] at page 344.

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* This term, "imaginarity," is not employed by English algebraists: its equivalent, *imaginarité*, is, however, of frequent occurrence in French works, and deserves to be imported into our language.

It may be well to give a specific name to pairs of imaginary roots of this class: we shall call them *primary pairs;* and the conditions [3] or [5], by which their entrance into an equation is discovered, are merely the embodiment in formulæ of these verbal statements, namely :

For a primary pair to exist, either, 1st, the second term of the proposed equation must vanish (for the proper transformation) between like signs; or, 2nd, the second term in the reciprocal of the proposed, or of some one or more of the derived equations, must vanish between like signs.

Of course, as already proved at (7), when the second term in the direct primitive vanishes between like signs, that is, when the condition [1] has place, the second term will also vanish from every derived equation, down to the quadratic, inclusive. But the condition [2], supplied by the final triad of terms in the primitive, will not be transmitted to the first derived equation, since the last term A_0 will not enter that equation; but whatever intermediate triads in the primitive satisfy the conditions [3], these triads, one after another, must become final triads eventually, in some of the derived equations; and the conditions [3] being satisfied for the corresponding triads in the primitive, they must be satisfied for these also, as proved at (7). The second term, therefore, in the reciprocal of every such derived equation must vanish between like signs.

(10) Such are the peculiar circumstances exclusively and invariably attendant upon the entrance of primary pairs of imaginary roots into an equation. As to the number of such pairs, when two or more of the conditions [3] are satisfied, or as to whether more than one pair can ever be safely inferred, however many of these conditions are satisfied—these are points in reference to which nothing can be determined at this stage of the inquiry, except the fact that in a *cubic* equation, though both the triads furnished by its four terms should equally satisfy the condition of primary pairs, yet only one pair of imaginary roots can enter the equation. A single illustration of the fact, in an equation of higher degree than the third, will suffice to show that the fulfilment of consecutive conditions [3], how many soever, does not imply, of necessity, more than one pair of imaginary roots. Thus, take the equation

$$4x^4 - 9x^3 + 8x^2 - 4x + 8 = 0,$$

all the triads of which satisfy the conditions [3],* namely,

$$\begin{array}{llll} \text{1st.} & 2n \times 4 \times 8 > (n-1)9^2, & \text{that is,} & 256 > 243 \\ \text{2nd.} & 3(n-1)9 \times 4 > 2(n-2)8^2, & \text{,,} & 324 > 256 \\ \text{3rd.} & 4(n-2)8 \times 8 > 3(n-3)4^2, & \text{,,} & 512 > 48 \end{array}$$

and yet, only two of the four roots are imaginary; a real root lies between 0 and $\frac{1}{2}$. For diminishing each root by ·5, we have

$$\begin{array}{rrrrr} 1 & -\,2{\cdot}25 & +\,2 & -\,1 & +\,{\cdot}2(\,{\cdot}5 \\ & {\cdot}5 & -\,{\cdot}875 & {\cdot}5625 & -\,{\cdot}21875 \\ \hline & -\,1{\cdot}75 & +\,1{\cdot}125 & -\,{\cdot}4375 & -\,{\cdot}01875 \end{array}$$

the change of sign showing that a root lies between 0 and ·5. We shall find, upon trial, that the first figure of this root is ·4.

This example clearly enough shows that the fulfilment of the conditions [3], one after another, without interruption, by the successive triads of an equation, is no proof that more than a single pair of imaginary roots enter; one pair there necessarily must be; other pairs there may be, but there are not other pairs necessarily. If the absolute number in the foregoing equation had been 1·4, or any greater number, instead of the number ·8, all the roots would have been imaginary, although only two imaginary roots could have been indicated by the preceding tests; for the coefficients would then have satisfied the condition

$$(4A_4A_2 - A_3^2)\,A_0 > A_4A^2_1,$$

which has place only when all the roots of the equation are imaginary, as may be proved as follows:—

(11) Let the general equation of the fourth degree, with the last term positive, be put in the form

$$x^2(A_4x^2 + A_3x + A_2 - p) + (px^2 + A_1x + A_0) = 0.$$

Then, if p be taken equal to $A_1^2 \div 4A_0$, the second of these quadratic expressions, being a square, will be positive (or zero) for every real value of x; and the first will be always positive also, that is, all the roots of the equation will be imaginary, if

$$4A_4\left(A_2 - \frac{A_1^2}{4A_0}\right) > A_3^2,$$

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[*] If the fourth term were $+\,4x$, instead of $-\,4x$, the first and last triads would each still satisfy the conditions; but the middle triad would fail. Yet, as the imaginary roots indicated by the first triad are indicated in the positive region of the roots, and the imaginary roots indicated in the last triad would then be indicated in the negative region, we should know that the two pairs of roots are distinct. As it is, however, all the roots are indicated in the positive region.

$$\text{or, } 4A_4A_2 - \frac{A_4A_1^2}{A_0} > A_3^2,$$

$$\text{or, } (4A_4A_2 - A^2_3)A_0 > A_4A_1^2.$$

And all the roots will still be imaginary if the sign = replace the sign >, since *each* of the two component parts of the equation, in the above form, will then be a positive square.

(12) As implied in the title of it, one object of the present communication is to prove the truth of the rule proposed by Newton, in the *Arithmetica Universalis*, for determining the number of imaginary roots in an equation, whenever the coefficients of that equation have certain specified relations among themselves—the relations, in fact, which are among those exhibited in the conditions [3], at page 344. In those equations, in which no one of these conditions is satisfied by the coefficients, Newton's rule is of no avail; although, notwithstanding the non-fulfilment of any of the conditions adverted to, the equation may have even all its roots imaginary. Yet, restricted as it thus is to the class of roots which we have called primary pairs, when two or more of such pairs enter an equation, the rule will frequently enable us to detect their presence; while the criteria [3] alone, how many soever of them might be satisfied by the coefficients, could never assure us of the existence of more than a single pair. By the aid of the general theorem at (7), Newton's Rule may be demonstrated as follows:—

(13) Returning to the primitive equation [*I*], suppose we were to deduce from it the several limiting equations in order: we know that the coefficients A_0, A_1, A_2, &c., would disappear one after another, the leading coefficient, A_n, being the only one that would be retained at the end of the operation. Suppose, now, the order of the coefficients to be reversed, as well in the primitive as in each of these derived equations, and that then the series of limiting equations be deduced from each of these, till we arrive at a limiting equation of the third degree; then, leaving blanks for whatever numerical factors may have been introduced into the coefficients by this process of derivation, the derived cubic equations will be

$$\begin{array}{l}
[\quad]\,A_0x^3 + [\quad]\,A_1x^2 + [\quad]\,A_2x + [\quad]\,A_3 = 0 \\
[\quad]\,A_1x^3 + [\quad]\,A_2x^2 + [\quad]\,A_3x + [\quad]\,A_4 = 0 \\
[\quad]\,A_2x^3 + [\quad\;[\,A^3x^2 + [\quad]\,A_4x + [\quad]\,A_5 = 0 \;.\;.\;.\; [\mathit{II}] \\
\qquad\qquad\qquad\vdots \\
[\quad]A_{n-3}x^3 + [\quad]\,A_{n-2}x^2 + [\quad]\,A_{n-1}x + [\quad]\,A_n = 0.
\end{array}$$

Now, for the purpose in hand, we are not interested in knowing what the numerical quantities are which would correctly fill up these blanks; it is sufficient for this purpose that we know, from the property established at (7), that, if we were to apply to each of these cubics the criterion of imaginary roots, n being equal to 3, be the wanting numbers whatever they may, we should obtain the very same conditions [3] *which the original coefficients A_0, A_1, A_2, &c., supply; each condition* here being the same as that condition there, into which the same triad of original coefficients enters. *Here*, however, we see that the last triad of terms in any cubic always involves the same coefficients of [*I*] as the first triad in the cubic next following; so that, if the criterion of imaginary roots be satisfied by the last three terms of one cubic, it must be satisfied by the first three of the next, and *vice versâ*. As a cubic equation cannot have more than one pair of imaginary roots, it follows that the fulfilment of the condition by any two consecutive sets of three terms of the primitive equation implies, of necessity, but one pair of imaginary roots in that equation.

When, however, *each* of the two triads of a cubic equation indicates imaginary roots, the concurring indications imply a distinct peculiarity in the pair of roots thus indicated. The peculiarity is this, namely, that not only does their entrance cause the second term of the cubic equation to vanish between like signs, but the entrance of their reciprocals, in the reciprocal equation, causes also the second term of *that* equation to vanish between like signs; that is to say, the second term vanishes between like signs, whether the coefficients be taken in the order proposed or in the reverse order.

When the first triad of terms satisfies the condition of imaginary roots, and the second triad fails, the reciprocal pair is not thus indicated; when the second triad satisfies the condition, and the first fails, it is in the reciprocal equation alone that the second term vanishes between like signs. And it is, moreover, only when such evanescence takes place in the *reciprocal* equation that imaginarity is necessarily conveyed from the one cubic to the other; and not only in the contrary case, that is, when the final triad fails to satisfy the condition, is imaginarity not *necessarily* conveyed to the next cubic, but it cannot possibly be conveyed under any circumstances whatever. In no single instance is a pair of roots in a cubic imaginary, either primary or non-primary, merely in consequence of the final triad in the preceding cubic being what it is, unless that preceding triad itself satisfies the condition of imaginarity. This may be proved as follows:—

(14) Let the cubic equation be

$$A'_3x^3 + A'_2x^2 + A'_1x + A'_0 = 0.$$

in which the leading triad of terms, supplied by the final triad of the immediately antecedent cubic in [*II*], fails to satisfy the condition at [3]; and let it be transformed into

$$x^3 + px + q = 0.$$

by the removal of the first coefficient, and the second term; p will then be necessarily negative, by the hypothesis. Now it is known that

1. If $\left(-\frac{p}{3}\right)^3 > \left(\frac{q}{2}\right)^2$, the roots will all be real and unequal.

2. If $\left(-\frac{p}{3}\right)^3 < \left(\frac{q}{2}\right)^2$, two of the roots will be imaginary.

3. If $\left(-\frac{p}{3}\right)^3 = \left(\frac{q}{2}\right)^2$, the roots will all be real, and two of them equal.

The second of these conditions shows that, in the proposed hypothesis, the three coefficients, A'_3, A'_2, A'_1, can never alone suffice to introduce imaginary roots into the complete cubic, since that condition implies that a suitable value of q, and consequently of A'_0, is indispensably necessary to such introduction. Hence the final triad of one of the cubics [*II*] can never introduce imaginarity into the cubic next following, unless that triad itself satisfies the condition of imaginarity. On the other hand, whenever the triad does satisfy that condition, no value of the absolute term (or q) can ever prevent imaginary roots from entering the cubic to which that triad is transferred; that is, under this condition imaginarity is of necessity introduced by the former cubic into the latter, let q, or A'_0, in this latter cubic, be whatever it may: it may indeed be anything or nothing, p being necessarily positive; and on account of this sign of p it is, and on this account solely, that two of the roots are of necessity imaginary; the value or sign of q (and therefore of A'_0) having nothing at all to do with the matter.

(15) It thus follows that when in any cubic equation

$$A'_3\, x^3 + A'_2\, x^2 + A'_1\, x + A'_0 = 0,$$

the condition

$$3\ A'_1\, A'_3 < A'^2_2 \ .\ .\ .\ .\ [a]^*$$

has place, it is impossible that imaginary roots can enter independently of the value of A'_0; so that imaginarity can never be introduced into the cubic (if the absolute term A'_0 be arbitrary), whatever values we may give to the three leading coefficients, provided only the condition [*a*] is preserved. But when, on the contrary, the condition is

$$3\ A'_1\, A'_3 > A'^2_2 \ .\ .\ .\ .\ [b],$$

then it is impossible that imaginary roots can be excluded, let A'_0 take whatever value it may; that is, under the condition [*b*], imaginary roots must enter the equation independently of the value of A'_0. The first of these two conclusions is that to which attention is here more especially invited. It is indispensably necessary to the inference which it is our main object here to deduce, that it should be clearly seen that the transference of the final triad of any one of the cubics [*II*] to the position of leading triad of the cubic next following can never be a transmission of imaginarity, from the one to the other, if the triad satisfy the condition [a]; a truth which is indeed sufficiently obvious from the first of the formulæ at (14); for removing the middle term of the transferred triad, the resulting p is necessarily negative; and therefore, by merely altering the value of q (that is of A'_0), if it be found to need altering for the purpose, without meddling with the transferred triad, all the roots may always be made real; which could not be done if the transference of the triad ever involved conveyance of imaginarity. The conclusion, therefore, is irresistible, that if each of these two cubics has imaginary roots—the second, in virtue of its *final* triad alone, satisfying the condition of a primary pair—the two pairs must be entirely independent of one another: the entrance of the second pair cannot possibly be a consequence of the entrance of the first pair.

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* The general condition $2n\, A_{n-2} < (n-1)\, A^2_{n-1}$, obviously becomes, in the case of the cubic, the particular condition in the text; and it is further obvious, from the group of conditions [5], that whenever the exponent n is odd, the multipliers are each of them even, and therefore divisible by 2.

(16) Let now the first of these two cubics, taken from [*II*], be represented by $C=0$, and the second by $C_1=0$; the imaginary pair in the former by I, and the imaginary pair in the latter by I_1; then, as just seen, the entrance of I_1 into the cubic $C_1=0$ is not a consequence of the entrance of I into $C=0$, but is entirely independent of the entrance of I. Calling the two reciprocal equations from which these cubics have been respectively deduced $R=0$, and $R_1=0$, we may, by reversing the process by which C has been derived from R, derive this latter from the former, adding in at each reverse step that particular constant (or final term) which in the direct step was made to disappear. In the equation of the fourth degree, the result of the first step from C, there enters an imaginary pair—a primary pair, necessarily and exclusively dependent for its character as such upon the pair in $C=0$.

But the pair in $C_1=0$ is *not* dependent for its imaginarity on this pair in the biquadratic; for if it were, it would be dependent on the pair in $C=0$, which it is not. In like manner, the equation of the fifth degree, in the next reverse step, has an imaginary pair dependent on the before-mentioned pair in the preceding result, and therefore on the pair in $C=0$: the pair in $C_1=0$ is therefore equally independent of *this* pair; and so on throughout all the reverse steps up to $R=0$, that is, there is a pair of imaginary roots in $R=0$, of which the imaginary pair in $C_1=0$, derived from $R_1=0$, is independent.

But imaginary roots can enter $R_1=0$ only as a consequence of imaginary roots entering $R=0$; and imaginary roots can enter $C_1=0$ only as a consequence of imaginary roots entering $R_1=0$; and *therefore* only as a consequence of imaginary roots entering $R=0$.

But it was shown that the pair in $C_1=0$ does not enter as a consequence of that particular pair in $R=0$, of which the pair in $C=0$ (in the reverse process of derivation) is the source: hence the pair in $C_1=0$ must be the consequence of some *other* pair in $R=0$; which equation has therefore at least two pairs of imaginary roots. Consequently the primitive equation has at least two pairs of imaginary roots.

The particular imaginary pair in the equation $R=0$, here adverted to, is that pair the entrance of which is indicated by the evanescence of the *second* term of the equation $R=0$ between like signs: in other words, if r be the transforming factor by which the second term is removed, the imaginary pair, traceable to the pair in the cubic $C=0$, is indicated between $r-\delta$ and $r+\delta$; since it is the *leading* triad of R that fulfils the condition—the same condition, by (7), as that fulfilled by the leading triad of C. No doubt, in certain cases, other coefficients, besides the second coefficient, may also vanish between like signs for the same transformation (r), and other pairs of imaginary roots be indicated between $r-\delta$ and $r+\delta$. But, by taking account of only a single pair, in all circumstances—the pair, namely, that would necessarily be an imaginary pair, though any or all of the coefficients in R, after the leading triad, were changed—we restrict ourselves, as we ought, to that pair alone, the imaginarity of which is conveyed, in the reverse process, through all the intermediate equations, from $C=0$, up to $R=0$, regardless of whatever other imaginary pairs may be, as it were, picked up and absorbed into the equation in its progress towards completion from C to R. Whatever modifications *this* pair may undergo from changes in the coefficients after the third term, its character as an imaginary pair in $R=0$ is still preserved, and it continues throughout to be indicated between $r-\delta$ and $r+\delta$.

It will have been observed, that in the foregoing reasoning R, is regarded as derived from C (after restoring at each step the factor previously expunged) by the process of *integration*, as we may for the moment call it: and it is to be noticed that it is the *general* integral, at each step that owes a pair of imaginary roots to the entrance of the pair in $C=0$; in other words, that although the constant completing any integral be taken of any arbitrary value, even zero, a pair of imaginary roots—primary roots, resulting from the pair in $C=0$, must still enter. The constants actually introduced are each of assigned value, because a specific equation, $R=0$, with assigned coefficients, is ultimately to be deduced. The constants, added one after another, as the derivation (or integration) proceeds, may cause the introduction of additional imaginary pairs, as just noticed; but none of *these* pairs are traceable to the pair in $C=0$; and a pair traceable to the pair in $C=0$ would still enter each of the ascending equations, though no constants at all were introduced.

(17) From what has now been shown, we see—always bearing the general property (7) in mind—that the search after distinct and independent primary pairs in the equation [*I*] may be converted into a search after the independent primary pairs in the group of cubic equations [*II*]; for although, in applying the tests [3] to each of these cubics, n is equal to 3, yet that when the blanks in the coefficients are filled up, the conditions, unaffected by this lower value of the exponent, become the very same as those marked [3], in which n is the leading exponent of the equation from which these cubics have been derived. So far, therefore, as this inquiry is concerned, the group [*II*] effectually replaces the single equation [*I*], with this advantage, namely, that as the individual equations [*II*] are connected together so that the final triad of coefficients of one supplies the leading triad of coefficients

of the equation next following, we can readily see whether or not imaginarity is conveyed from the former to the latter. After the first triad of [*I*] (or the first when the coefficients are reversed, it makes no difference) each successive triad up to the last is thus repeated in [*II*] : it is this triad—common to two consecutive cubics, which forms the connecting link mentioned, and which causes the same fulfilment or failure of the condition [3], in the leading triad of the second of these cubics, as in the final triad of the first. And this is the only connecting link between the two : in other respects they are independent, so that when the final triad of a cubic (and consequently the leading triad of the next succeeding cubic) fails to satisfy the condition [3], and the final triad of this succeeding cubic does satisfy that condition, an imaginary pair, distinct from whatever other pair or pairs have been inferred from earlier cubics in the series, must enter the primitive equation: and this is the same as saying that when a fulfilment of a condition [3] by a triad of the coefficients of [*I*] is preceded by a failure, a pair of primary roots, distinct from and independent of whatever pairs may previously have been detected, is indicated in the equation.

(18) But before the passage from a fulfilment to a failure, or from a failure to a fulfilment, there may have been a continuous succession of such fulfilments or failures in passing from cubic to cubic; or, which is the same thing, in proceeding from term to term of the primitive equation. From these uninterrupted concurrences we cannot infer anything, as to additional imaginary pairs: such additional pairs *may* enter the primitive, or they may not; as is sufficiently exemplified in article (10): but we shall always be on the safe side—that is, we shall never be in danger of inferring *more* pairs than really enter—if we always regard these concurrences as merely repeated indications of one and the same thing, namely, the succession of fulfilments as only so many concurring proofs of the existence of but one pair of imaginary roots, and the succession of failures as indications that no additional imaginary pair is to be inferred so long as the failures remain uninterrupted by a fulfilment.

We may remark, however, that when there is a continuation of fulfilments, a peculiar character is impressed upon the several cubic equations [*II*], as already adverted to at (13) : the triad supplied to a cubic by the antecedent cubic, imports primary imaginarity simply; whilst the triad which the new fourth term completes, so modifies the roots that, whether we take the direct or the reciprocal equation, the second term in either case vanishes between like signs; and imaginarity cannot be expelled, whether we change the final term or the leading term; it is what it is in virtue of *both* triads satisfying the condition independently; and is as much a consequence of one triad as of the other. Independent imaginarity in any one of the cubics after the first cubic can be inferred only when the imaginarity is due exclusively to the final triad, and may therefore be expelled from the equation by merely modifying the final term, that is, the absolute number which completes the cubic. Of course it will be understood throughout these remarks that transference, or conveyance, of primary imaginarity always implies transference of a *triad* of terms.

(19) From what has now been established, we deduce the following rule for determining (at least approximately) the number of imaginary roots in a numerical equation, from the mere examination of the coefficients.

Rule 1. Under the leading term of the equation, write the sign *plus*, as the first of a row of signs.

2. Then, taking the second coefficient of the equation as the middle one of the first three coefficients, apply to those three the proper test [3] or [5]. If the condition be satisfied, write *minus* under the second term; if it be not satisfied, write *plus*. In other words, plus or minus is to be written under the second coefficient according as its square, multiplied by the proper factor, is greater or less than the product of the adjacent coefficients multiplied by the proper factor.

3. Passing to the third coefficient; take *that* as the middle of the two adjacent coefficients; and apply, in like manner, the next following test; and as before, annex to the former signs *minus*, or *plus*, according as the condition holds or fails. And in this way proceed till all the coefficients have been employed. Then as many changes as there are in this completed row of signs, from *plus to minus* (not from minus to plus), so many pairs of imaginary roots must enter the equation: it may have more pairs, but it cannot have fewer.

Note.—The last triad of coefficients need not be tested whenever the row of signs already written down terminates in a minus sign; and it is well to remember that the test for the last triad is always the same as that for the first; for the last but one, the same as that for the second; and so on.

It may further be observed that it is impossible for fulfilments in the positive region of the roots to be succeeded by fulfilments in the negative region, or for fulfilments in the negative region to be succeeded by fulfilments in the positive region, without a separating failure; for whether permanencies of sign in the equation, are succeeded by variations, or variations by permanencies, the sign which is the termination of the one set, and the commencement of the other set, must evidently always have the sign adjacent to it on the one side of opposite character to that adjacent to it on the other side; so that the sign written under the middle one of the three must always be +. The region, therefore, in which a pair of imaginary roots lies, or in which the pair is indicated, is sufficiently marked by the collocation of signs in the equation.

(20) The following are applications of the foregoing rule:—

1.
$$\begin{array}{cccccc} x^5 & -\,4x^4 & +\,4x^3 & -\,2x^2 & -\,5x & -\,4 = 0 \\ + & + & - & + & + & \end{array}$$

The equation has one pair of imaginary roots, at least. By the rule o Descartes, there are but three roots in the positive region; two of these are those here found to be imaginary.

2. $$3x^5 + 8x^4 - 5x^3 + 12x^2 - 7x + 9 = 0$$
$$+ \quad + \quad + \quad -$$

The last triad here need not be tested, as the row of signs already written down terminates in a minus sign. One pair of imaginary roots is detected in the positive region.

3. $$7x^4 - 2x^3 + 8x^2 - 5x + 17 = 0$$
$$+ \quad - \quad + \quad -$$

In this equation all the roots are imaginary.

In applying the tests it will always be found preferable to employ them in the forms marked [5] at page 344. The multipliers in the right-hand members of these forms are all included in the general expression $m\ (n - m)$, and those in the left-hand members, are these each increased by the constant number $n + 1$. In an equation of a high degree, the easiest way of proceeding will be to place under the second term of the equation $n - 1$, under the third term $2\ (n - 2)$, under the fourth term $3\ (n - 3)$, and so on; in other words, commencing at the second term, to multiply the exponents by 1, 2, 3, &c., placing the results underneath; remembering that these numbers will recur in reverse order when the middle term, or the first of the two middle terms, is reached: each will be the multiplier for the square of the term under which it is placed; and when this multiplier is increased by the constant number $n + 1$, that is, by the leading exponent *plus* 1, the result will be the multiplier for the product of the two extreme terms of the triad we are testing: thus,

4. $$5x^8 - 2x^7 + 3x^6 - 24x^5 - 16x^4 + x^3 - 4x^2 - 2x - 60 = 0$$
$$7 \quad 12 \quad 15 \quad 16 \quad 15 \quad 12 \quad 7$$

These numbers are the multipliers for the squares of the coefficients immediately above them; and those for the product of the extreme coefficients of the triad, found by adding 9 to each number, are—

$$16 \quad 21 \quad 24 \quad 25 \quad 24 \quad 21 \quad 16$$

or, expunging common factors, the two rows of numbers will be those here underwritten:

$$5x^8 - 2x^7 + 3x^6 - 24x^5 - 16x^4 + x^3 - 4x^2 - 2x - 60$$

For the squares,

$$7 \quad 4 \quad 5 \quad 16 \quad 5 \quad 4 \quad 7$$

For the products,

$$16 \quad 7 \quad 8 \quad 25 \quad 8 \quad 7 \quad 16$$

And since

$$16.5.3 < 7.2^2,\ 7.2.24 > 4.3^2,\ -8.3.16 < 5.24^2,\ -25.24.1 < 16.16^2,$$
$$8.16.4 > 5.1^2,\ -7.1.2 < 4.4^2,\ \text{and}\ 16.4.60 > 7.2^2,$$

the row of signs will be

$$+ \quad + \quad - \quad + \quad + \quad - \quad + \quad -$$

so that the equation has six imaginary roots. As the last term of the equation is negative, the two remaining roots are real—one positive, and the other negative.

It may be observed here that when the extreme terms of any triad have unlike signs, as is the case with the third, fourth, and sixth of the triads above, it may always be passed over; the corresponding sign in the row of signs being written +, to imply that the triad in question fails to satisfy the condition of imaginary roots.

It may be remarked, too, that the accurate calculation of the two members of the inequality appealed to is but seldom necessary. Which of these two members is in excess of the other may, in most instances, be ascertained at a glance.

In the foregoing discussion, the case in which the sign of inequality in one or more of the criteria of imaginary roots is replaced by the sign of equality is not adverted to, except in so far as to notice (6), that one pair at least is then implied. There is another case, too, namely, that in which consecutive zeros occur among the terms of equation, which has not been specially considered above. The two cases have a certain relation to each other, and it remains for us to examine what are the inferences which these peculiarities justify. We shall first consider the case of consecutive zeros.

(21)—1. *When there are consecutive zeros.* If *all* the terms between the first term and the last are zeros, as in the equation

$$A_n x^n + 0 + 0 + 0 + \ldots\ldots + 0 + A_0 = 0 \ldots [1]$$

the exact number of imaginary roots is determinable at once; for since then

$$x = \left(-\frac{A_0}{A_n}\right)^{\frac{1}{n}},$$

it is obvious that if n be even, and A_0 negative, there will be just two real roots, numerically equal, but of opposite signs; and, therefore, $n-2$ imaginary roots; while if A_0 be positive, all the roots will be imaginary.

But if n be odd, then, whether A_0 be negative or positive, there will be only one real root; and, therefore, $n-1$ imaginary roots.

The terms of an equation of degree n, being $n+1$ in number, the foregoing zeros are $n-1$ in number. Hence, if we apply the rule at page 355 to the equation [1] above, for the purpose of marking the number of imaginary pairs, we must evidently write *minus* under the first zero; and then, to secure conformity with the foregoing conclusions, must write the signs *plus* and *minus* alternately, till the last zero is reached, the sign under which, if n be even, must always be the opposite of the sign of A_0; but if n be odd, this sign may be + or − indifferently.

Thus—

$$\textit{When } n \textit{ is even, } \left\{ \begin{array}{l} A_n x^n + 0 + 0 + 0 + \ldots . + 0 \pm A_0 \\ \quad + \quad\; - \quad\; + \quad\; - \qquad\qquad\; \mp \end{array} \right.$$

$$\textit{When } n \textit{ is odd, } \left\{ \begin{array}{l} A_n x^n + 0 + 0 + 0 + \ldots . + 0 \pm A_0 \\ \quad + \quad\; - \quad\; + \quad\; - \qquad\qquad - \text{ or } +\text{, indifferently.} \end{array} \right.$$

For taking account of the leading sign + in the underwritten row of signs, it is readily seen that the $n - 1$ zeros, when n is even, furnish $\frac{n}{2}$ changes from + to −, if A_0 is +, and $\frac{n}{2} - 1$ changes if A_0 is −. And that when n is odd, the number of changes is $\frac{n-1}{2}$, whether A_0 is + or −: and since each change from + to − implies a distinct pair of imaginary roots, the number of such roots, thus indicated, is precisely the same in each case as the number determined above. As the sign under the zero immediately preceding the last zero is always + when n is even, and always *minus* when n is odd, in the latter case it is plainly matter of indifference which sign be placed under the *last* zero.

But suppose that two or more significant terms precede or follow the zeros, or both precede and follow. By taking the successive limiting equations, these latter terms will disappear one by one, till only a single significant term beyond the zeros is left; and by reversing the coefficients of the equation thus reached, and proceeding in like manner, we shall finally arrive at an equation of the form

$$A'_{n'} x^{n'} + 0 + 0 + 0 + \ldots + 0 + A'_0 = 0 \ldots [2],$$

that is, at an equation of the same form as the equation [1] above.

(22) If, in this latter equation, $A'_{n'}$ should be positive, like A_n in [1], the foregoing conclusions, as to the number of imaginary roots, would of course apply to it; but if $A'_{n'}$ be negative, we should have to change the extreme signs of [2], or to multiply the terms by − 1, before we could deduce the number of imaginary roots, as above, from the underwritten signs. Yet, leaving the leading minus sign and the sign of A'_0 unchanged, if, as before, we write + under the first term, and − under the first zero; then + under the next, and so on, as directed above, till we come to the last zero, and write under that + or −, according as A'_0 is + or −, it is easy to see that the underwritten row of signs will have the same changes from + to − as if the signs of $A'_{n'}$ and A'_0 had themselves been changed; for it will be remembered that, in the case of n' being odd the changes from + to − are the same, whichever of these signs be placed under the last zero. Now, since the signs of $A'_{n'}$ A'_o, are the same as the signs of the original coefficients from which they have been derived, and as the intervening zeros are the same in number as in the original equation, it is plain that actual derivation is not necessary. All we have to do is to proceed with the

underwritten signs in conformity with the rule at page 355, till we come to that term which immediately precedes the zeros; under this (whatever be its sign) to write +; under the first zero, −; and so on alternately, till we reach the last zero, under which is to be written +, if the signs of the two terms which bound the zeros are unlike, and − if they are like, and then to proceed according to the rule. The sign under the term which immediately follows the zeros, as well as that under the term which immediately precedes them, will, of course, always be +.

NOTE.—It was directed above that when the first of the boundary-terms is *minus*, the sign of the other boundary-term is that which is always to be written under the last zero. If it should be for the moment thought that what has just been said is inconsistent with that direction, the reader has only to reflect that, in the case of $A'_{n'}$ negative, if A'_0 be +, the signs of the two terms will be *unlike*, and that if A'_0 be −, that the signs will be *like*; so that in the former case, + is to be written, and in the latter case −, the sign being always (in the case of $A'_{n'}$ negative), the same as the sign of A'_0.

We shall now give some examples of the application of these precepts.

1.
$$\begin{array}{l} x^5 + ax^4 + 0 + 0 + 0 + e = 0; \\ +\quad\;\; +\quad\;\; -\quad +\quad - \end{array}$$

2.
$$\begin{array}{l} x^5 + ax^4 + 0 + 0 + 0 - e = 0 \\ +\quad\;\; +\quad\;\; -\quad +\quad + \end{array}$$

The first of these equations has four imaginary roots; the second two, at least; of the other three, one is positive, and the remaining two belong to the negative region. [It may be observed here that if each of the roots of an equation be diminished by δ, the number δ may obviously be taken so small, that in carrying on the transforming process, by Horner's method, each addend may be made as small as we please; so small, therefore, that the signs of the significant terms of the original shall all be preserved unaltered in the transformed equation; in which case, what was a zero in the original, will, in the transformed equation, be a significant term, with the same sign as the significant term immediately preceding the zero. There will thus be a *permanence* of sign; and in this way permanencies will replace the arbitrary signs of all the zeros.]

3.
$$\begin{array}{l} x^7 - 2x^6 + 3x^5 - 2x^4 + x^3 + 0 + 0 - 3 = 0 \\ +\quad\;\; -\quad\;\; +\quad\;\; -\quad\;\; +\quad -\quad + \end{array}$$

Therefore six of the roots are imaginary.

4.
$$\begin{array}{l} x^7 - 2x^6 + 3x^5 - 2x^4 - x^3 + 0 + 0 - 3 = 0 \\ +\quad\;\; -\quad\;\; +\quad\;\; +\quad\;\; +\quad -\quad - \end{array}$$

Here the equation has four imaginary roots, at least; one real root is positive; the other two roots are doubtful, and belong to the negative region.

5. $$5x^8 - 2x^7 + 3x^6 + 0 + 0 + 0 + 4x^2 - 2x + 60 = 0$$

$$+ \quad - \quad + \quad - \quad + \quad - \quad + \quad -$$

Therefore all the roots are imaginary. This conclusion may be verified as follows. Writing the equation thus:—

$$x^6(5x^2 - 2x + 3) + (4x^2 - 2x + 60) = 0,$$

we see that each of the two quadratic expressions is positive whatever real value be given to x; and, therefore, as x^6 is always positive too, no real value of x can satisfy the equation.

6. $$3x^6 - 5x^5 + 0 - 2x^3 + 7x^2 + 0 - 4 = 0$$

$$+ \quad + \quad - \quad + \quad + \quad +$$

Consequently, only two imaginary roots are detected; they are in the negative region.

7.*

$$x^{16} + 0 + 0 + 0 + 0 - ax^{13} + 0 + 0 + 0 - bx^9 + cx^8 + 0 + dx^6 + 0 + 0 + 0 + 0 + 0 - e = 0$$

$$+ \quad - \quad + \quad - \quad + \quad \pm \quad - \quad + \quad - \quad + \quad + \quad - \quad + \quad - \quad + \quad - \quad + \quad +$$

Therefore, the equation has fourteen imaginary roots, at least; there may be sixteen, but there cannot be a greater number, since, as the sign of the last term shows, two roots, at least, must be real. If this sign had been +, then the sign under the last zero would have been −; and the equation would have had eight pairs of imaginary roots at least.

(23) 2. *When there are triads of equality.* Let us first suppose that all the triads, except the last triad, furnish conditions of equality. We except the last triad, because if every triad throughout were to satisfy the condition of equality, the roots would all be real and equal; that is to say, the equation would be of the form

$$A_n(x + r)^n = 0;$$

so that

$$x = -r = -\frac{A_{n-1}}{nA_n}$$

would express each of the n roots. But if, as here supposed, the last term A_0 of the equation be of such value as to render the final triad one of inequality—all the preceding being triads of equality—then it is plain that the form of the equation will be

$$A_n(x + r)^n + c = 0;$$

or

$$(x + r)^n = -\frac{c}{A_n}$$

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* From Fourier—"Analyse des Equations Déterminées."

(c being positive or negative according as

$$A_0 = A_n r^n + c, \quad \text{or} \quad A_0 = A_n r^n - c.$$

In the former case A_0 is in excess, in the latter case, in defect)

$$\therefore x = -r + \left(-\frac{c}{A_n}\right)^{\frac{1}{n}}$$

which shows that when n is even, and, moreover, c positive, all the n roots will be imaginary; but, if in this case of n even, c be negative, only $n - 2$ of the roots will be imaginary. And that when n is odd, then whether c be positive or negative, x will have one real value, and one only. From this it appears that—

When n is even, and it be found necessary to ***subtract*** a positive number (c) from A_0 to make the triad one of equality, the equation will have all its n roots imaginary; but if it be necessary to *add* a positive number (c) to A_0 for this purpose, then the equation will have $n - 2$ imaginary roots, and no greater number.

When n is odd, the equation will have $n - 1$ imaginary roots, whatever be the value of A_0, or, which is the same thing, whether c be subtractive or additive.

(24) The conclusions, then, are quite analogous to those deduced above from the case in which the $n - 1$ terms between $A_n x^n$, and A_0, are zeros, instead of significant terms having the peculiar relations to one another here supposed. *If n is even*, and A_0 in *excess*, the sign to be placed under the last of these intervening terms is to be $-$; and if A_0 is in *defect*, the sign is to be $+$; but *if n is odd,* then, in the case we are considering, as in the case of the zeros, it is matter of indifference whether the sign under the last of the intervening terms be $+$ or $-$; the number of imaginary roots indicated being the same, whichever sign be chosen. For the immediately preceding sign in the row will always be $-$, inasmuch as the sign under the first of the odd number of intervening terms is itself $-$; and the underwritten signs are alternately $-$ and $+$. But, having in view the case to be considered in the article next following, and in order to preserve uniformity in both cases, it will always be better to write $+$ or $-$, according as A_0 is in defect or in excess, just as in the case of n being even.

The foregoing conclusions may be arrived at in another way. Putting the proposed equation in the form given to it above, namely, $A_n (x + r)^n + c = 0$, and supplying the $n - 1$ zero-terms, it becomes

$$A_n(x + r)^n + 0 + 0 + 0 + \ldots + 0 + c = 0,$$

which, $x + r$ being here in the place of x, is identical with the form [1], at page 357, and whatever values, real or imaginary, x has in the former equation, so many must $x + r$ have in this, and *vice versâ*, since the value of r is always real.

Suppose, now, that triads of equality occur *anywhere* among the terms of an equation; then, by taking the successive limiting equations, as in the case of consecutive zeros, we can reach an equation in which all the triads are triads of equality, except the last triad,* and can thus return to the case just considered. And it is plain, from what has already been shown, that the signs to be written under the terms which intervene between the first term and the last term of this derived equation will be alternate signs, like as if these intervening terms were so many zeros; and, as in the case of the zeros, these are the signs to be written under those terms of the primitive equation from which the terms here spoken of, in the derived equation, have been deduced. But one thing must be attended to here, which, in the case of intervening zeros, requires no special notice. The signs to be written under the first term of the leading triad, and under the last term of the series of triads we are here considering, are not necessarily +, as in the case of the zeros, but may be either both −, or one + and the other − : *which* sign is to be underwritten, the general rule at (19) will, of course, enable us to readily ascertain. If the former of these two signs is seen to be −, the sign immediately next following must be +; and so on, alternately, till the latter of the two is, in like manner, determined by the rule, and underwritten. In the case of the zeros, the first of these two signs was invariably + (and so was the last); and the immediately next sign, −; but it may be otherwise here, as the general rule must be obeyed. It may be observed, however, that an underwritten − always implies a pair of imaginary roots, as a + must have preceded it; for it is with a + that the row of signs under the terms of the proposed equation commences; so that no *fewer* imaginary pairs can ever be indicated by the signs under the terms of the primitive than would be indicated by the signs under the terms of the final derived equation.

We shall now give an example or two by way of practical illustration :—

1. $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 36 = 0$

$+ \quad - \quad + \quad - \quad -$ [Here $A_0 = 36$, is in excess.]

Hence there are four imaginary roots.

2. $16x^4 - 96x^3 + 216x^2 + 216x - 80 = 0$

$+ \quad - \quad + \quad +$ [Here $A_0 = -80$, is in defect.]

So that the equation has but two imaginary roots.

3. $x^8 + 5x^7 - 3x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 6x + 4 = 0$

$+ \quad + \quad - \quad + \quad - \quad + \quad + \quad -$

[Here the 4 is in excess.]

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* If the last triad satisfy the condition of equality, and not all the triads, then, just as in the case of the first triad being a triad of equality, an imaginary pair will be indicated by that triad alone (Art. 6).

Consequently there are at least six imaginary roots, two in the positive region, and four in the negative region.

4. $$3x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 - 8x^2 + 12x - 5 = 0$$

$$+ \quad - \quad + \quad - \quad + \quad - \quad +$$

Hence there are six imaginary roots in the equation.

As in the first and fourth of these examples the sign immediately preceding the last is *minus*, we might, in each of these, have stopped at that sign: the determination of the *last* sign was unnecessary, as no additional imaginary roots could be indicated, whether the last sign proved to be + or −; nor could another pair have been indicated, though the degree of the equation had been even instead of odd.

(25) We here terminate these practical details respecting Newton's Rule, which rule is substantially the same as that given at page 355 of this paper. We have not attempted any *extension* of it, but have been content with ascertaining what is the utmost amount of information, respecting the number of imaginary roots in a numerical equation, that can be educed from it. The rule itself does not appear to the present writer as capable of any extension—if by that term be meant its being so enlarged as to be available for detecting the presence of imaginary roots other than those which, in the foregoing investigations, have been called *primary pairs*.* Before, however, passing to other matters, it may be well to give a practical illustration of the way in which we may always ascertain whether or not any proposed polynomial is the development of the binomial form $A_n(x + a)^n$: it was adverted to at p. 346.

Suppose, for instance, we wished to know whether or not the polynomial following is a binomial development. (See p. 356.)

$$27x^6 - 108x^5 + 180x - 160x^3 + 80x^2 - 21\tfrac{1}{3} + \frac{64}{27}$$

Mult. for the squares,	5	8	9	8	5
,, ,, products,	12	15	16	15	12

As we find that all the triads satisfy the conditions of equality, and that here

$$a, \text{ or } -\frac{A_{n-1}}{nA_n}, \text{ is } \frac{108}{6 \times 27}, \text{ or } \frac{2}{3},$$

we infer that the invelopment of the proposed polynomial is $27\left(x - \frac{2}{3}\right)^6$. Of course if the last term of any polynomial, when divided by A_n, the coefficient of the first term, be not a complete power (a^n), the root of which is $a = -A_{n-1} \div nA_n$, we should at once know that the polynomial cannot be the development of any expression of the form $A_n(x \pm a)^n$; nor can it be if the terms be neither all positive, nor yet alternately positive and negative. Yet if any number of consecutive triads satisfy the conditions of equality, an expression of this form may always be found by computing towards the left, as well as towards the right (if the consecutive triads be intermediate terms), such that those terms shall be identical with the corresponding terms of the development for *some* values of A_n and a. [The foregoing method of computing term by term, may of course be employed for developing any case of $A_n(x \pm a)^n$].

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* It will be hereafter shown, however, that non-primary pairs in an equation are always convertible into primary pairs by diminishing each of the roots by a determinable number.

(26) The remainder of the present communication will be quite independent of the rule of Newton, and of everything that has preceded, except the group of Criteria at page 344. These formulæ [3] are merely deductions from the principle of De Gua; but as we shall have frequent occasion to advert to the numerical multipliers connected with the formulæ [3], and as these same multipliers are those employed in the rule of Newton, we shall for brevity and convenience refer to them under the denomination of Newtonian factors. The following general property will be found useful in the business of actual solution.

If an equation be represented by the notation $f(x) = 0$, and its roots be each diminished by r, the transformed equation will be $f(x + r) = 0$, each of the roots (x) of which will be less by r than the corresponding root (x) of the original equation:* and we know from the theory of equations, that $f(x + r)$ may be written either

$$\left.\begin{aligned} f(x+r) &= f(x) + f_1(x)r + \frac{1}{2}f_2(x)r^2 + \frac{1}{2.3}f_3(x)r^3 + \dots + \\ &\qquad\qquad \frac{1}{2.3.4\dots n}f_n(x)r^n = 0 \\ \text{or} \qquad & \\ &= f(r) + f_1(r)x + \frac{1}{2}f_2(r)x^2 + \frac{1}{2.3}f_3(r)x^3 + \dots + \\ &\qquad\qquad \frac{1}{2.3.4\dots n}f_n(r)x^n = 0 \end{aligned}\right\} [1]$$

Now the property we propose to prove is this, namely:—If the middle one of any three of the consecutive functions,

$$f(x),\quad f_1(x),\quad \frac{1}{2}f_2(x),\quad \frac{1}{2.3}f_3(x),\ \dots \qquad [2]$$

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* Of course, although we here speak of the roots being "diminished" by r, it will be understood that r may be regarded as either positive or negative. Indeed the property in the text holds whether r be real or imaginary, as the general demonstration of it proves.

be squared, the first two terms of the result, when multiplied by the proper Newtonian factor (as suggested by the degree n of the equation), will always be the same as the first two terms of the product of the extreme terms when multiplied by the corresponding Newtonian factor.

Let any consecutive three of the functions [2] be represented by

$$Ax^p + A'x^{p-1} + \dots\,.$$

$$\frac{1}{m}pAx^{p-1} + \frac{1}{m}(p-1)A'x^{p-2} + \dots\,.$$

$$\frac{1}{m(m+1)}p(p-1)Ax^{p-2} + \frac{1}{m(m+1)}(p-1)\,(p-2)A'x^{p-3} + \dots\,.$$

The first two terms of the product of the first and third of these expressions retaining coefficients only, are—

$$\frac{1}{m(m+1)}p(p-1)A^2 + \frac{1}{m(m+1)}\{p(p-1) + (p-1)(p-2)\}AA'$$

or
$$\frac{1}{m(m+1)}p(p-1)A^2 + \frac{2}{m(m+1)}(p-1)^2AA'\dots \qquad [3]$$

and the first two terms of the middle expression squared, coefficients only being retained, are—

$$\frac{1}{m^2}p^2A^2 + \frac{2}{m^2}p(p-1)AA'\dots \qquad [4]$$

By multiplying [3] by $\frac{m+1}{m}\cdot\frac{p}{p-1}$, we get [4]: therefore if we multiply the former expression by $(m+1)p$, and the latter by $m(p-1)$, the results will be the same; and these two multipliers are the proper Newtonian factors, as is easily seen by putting 1, 2, 3, &c., in succession for m, and n, $n-1$, $n-2$, &c., in succession for p.

The same conclusion may be arrived at more expeditiously thus—

If $f(x+r)$ were a *power*, that power would, of course, be

$$(x+r)^n = \left(x + \frac{A_{n-1}}{nA_n}\right)^n = x^n + \frac{A_{n-1}}{A_n}x^{n-1} + \dots \qquad [5]$$

and the triads of [1] would all be triads of equality. The square of a middle term, multiplied by its Newtonian factor, would be a result which, in *all* its terms, wouldbe the same as the product of the extreme terms, multiplied by the proper factor. But though $f(x+r)$ be *not* equal to $(x+r)^n$, $f(x)$, and, therefore, all the functions [2] derived from it will have the *first two* terms of each the very same as if the polynomial were the power [5], as is obvious; and hence the truth of the theorem announced above.

(27) An immediate deduction from this theorem is, that if the condition of imaginarity hold or fail for the first three coefficients of any equation, it will, in like manner, hold or fail for the first three coefficients of every transformed equation which can result from increasing or diminishing the roots by any quantity (r) whatever.

For the proposed equation being

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots + A_0 = 0,$$

we know that the first three coefficients of the transformed equation $f(x + r) = 0$ will be the last three of the second development [1]; and, therefore, writing the final coefficient first, that they will be of the forms

$$A_n, \quad ar + A_{n-1}, \quad br^2 + cr + A_{n-2}.$$

But, by the foregoing general principle, if the condition hold or fail for the three original coefficients, A_n, A_{n-1}, A_{n-2}, it must equally hold or fail for these, inasmuch as that $2n$ times the product of the first and third of them differs from $n - 1$ times the square of the second by exactly the same amount that $2nA_nA_{n-2}$ differs from $(n-1)A^2_{n-1}$; for the two terms involving r disappear from the difference, as just proved.

Hence, for every transformation, these two functions of the coefficients have the same constant difference.

Suppose we had developed one of the roots of a cubic equation by Horner's method, and that we wished to ascertain whether the roots of the quadratic equation

$$A_3 x^2 + A'_2 x + A'_1 = 0,$$

to which the process would conduct us, were real or imaginary; that is, whether $4A'_1A_3$ be less or greater than A'^2_2. Now we know, from the foregoing principle, that

$$3A_1A_3 - A_2^2 = 3A'_1A_3 - A'^2_2 \ldots . \qquad [6]$$

and, therefore, having calculated the first of these expressions from the original coefficients, we see that we have only to ascertain whether $A'_1 A_3$ added to it will make the result positive or negative. If positive, the other two roots of the cubic equation are imaginary; if negative, they are real. For example: In the writer's treatise on "The Analysis and Solution of Cubic Equations" (p. 172), the development of a root of the equation

$$x^3 + 8x^2 + 6x - 75{\cdot}9 = 0$$

conducts to the quadratic

$$x^2 + 15{\cdot}2771x + 62{\cdot}46326147 = 0$$

and from the principle above, we readily see, without any calculation with these large numbers, that since

$$18 - 64 + 62\cdot \ldots \text{ is positive,}$$

the remaining roots of the cubic must be imaginary.

But it may be well to show here the amount of numerical labour spared when the character of the roots, from close proximity to equality, is much less readily discoverable, by the ordinary method, than in this example.

By the method here proposed, the work is

$$\begin{array}{r} 18 - 64 = -\ 46 \\ \underline{62\cdot 46326147} \\ \underline{16\cdot 46326147} \end{array}$$

By the common rule (after multiplying the absolute number by 4),

$$\begin{array}{rl} & 249\cdot 85304588 \\ (15\cdot 2771)^2 = & \underline{233\cdot 38978441} \\ & 16\cdot 46326147 \end{array}$$

If the first of the expressions [6] be itself positive, we should know, at the outset, that two roots are imaginary. If it be zero, then, since, when increased by a positive quantity, the result would be positive, we should know then, also, at the outset, that two roots would be imaginary; and similarly in reference to the second triad of coefficients, as the coefficients may be reversed. But both of these conclusions may be inferred from what has been previously established.

(28) Returning now to the first of the equations, p. 364, if we represent the original polynomial $f(x)$ by X_0, and the successive coefficients of r, r^2, &c., in [1], that is, the several derived polynomials, by X_1, X_2, &c., respectively, a very general form may be given to the criteria of imaginarity at p. 344, namely,

$$\begin{array}{rcl} 2nX_0X_2 & > & (n - 1)X_1^2 \\ 3(n-1)X_1X_3 & > & 2(n-2)X_2^2 \\ 4(n-2)X_2X_4 & > & 3(n-3)X_3^2 \\ 5(n-3)X_3X_5 & > & 4(n-4)X_4^2 \\ & \vdots & \\ 2nX_{n-2}X_n & > & (n-1)X^2_{n-1}\text{*} \end{array}$$

in which expressions the x involved in X_0, X_1, &c., may take any real value, positive or negative, whatever. If we put $x = 0$, the formulæ become those at page 344; X_0, X_1, &c., and A_0, A_1, &c., then being identical. Take, for example, the equation

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* X_n is, of course, always identical with A_n.

$$\begin{aligned}
&X_0 = 4x^4 - 10x^3 + 9x^2 - 3x + \tfrac{1}{4} = 0\\
\text{then } &X_1 = 16x^3 - 30x^2 + 18x - 3\\
\text{,, } &X_2 = 24x^2 - 30x + 9\\
\text{,, } &X_3 = 16x - 10\\
\text{,, } &X_4 = 4
\end{aligned}$$

When $x = 0$, these give the original coefficients, namely,

$$\begin{array}{ccccc} X_4 & X_3 & X_2 & X_1 & X_0 \\ 4 & -10 & +9 & -3 & +\tfrac{1}{4} \end{array}$$

and for underwritten signs, + + + + , showing no imaginary roots.

But when $x = 1$, the results are—

$$\begin{array}{ccccc} X_4 & X_3 & X_2 & X_1 & X_0 \\ 4 & +6 & +3 & +1 & +\tfrac{1}{4} \end{array}$$

and the underwritten signs, + + − , showing two imaginary roots.

The other two roots are found to be real; one lies between ·1 and ·2, and the other is ·5.

The coefficients last written are those of that transformed equation, which we should get by diminishing each root of the original equation by 1—an operation much more readily performed than that above. And we see, by this example, that non-primary pairs of imaginary roots in an equation may become primary pairs in the transformed equation that would result from diminishing or increasing each root by some number; but what this number is we can discover only by trial, or by a previous analysis of the equation. Yet, by developing a real root by Horner's method of approximation, whenever we find that two variations of sign are lost or gained, in passing from one step of the operation to the next, we may always ascertain, as here, whether the two roots passed over are certainly imaginary, or possibly real; and in thus testing the several triads in any of the transformed equations, the first triad need never be examined, since, by the foregoing theorem, the results of such examination is already known from the three leading coefficients of the primitive equation. And, as regards other triads, any nice calculation of the squares and products of the coefficients will very seldom be necessary; a mere glance at those coefficients will often suffice to assure us whether the criterion of imaginarity is satisfied or not. We can generally see, by inspection, whether the square of the middle coefficient of any triad is less (or not greater) than the product of the other two coefficients; and, since the numerical multiplier of the square is always less than the numerical multiplier of the product, we may thus, in most instances, have sufficient indication of the presence of a pair of imaginary roots without any actual numerical work.

(29) We have stated above that in developing a real root by Horner's method, it would be well, so soon as the two variations are

lost or gained in a transformed equation, to apply the test of imaginarity anew; but we need not wait till such changes occur : the test will be as likely to discover the presence of an imaginary pair before any change of variations takes place, as after; so that when in the case of doubtful roots in the assigned interval, we carry on the work of approximation in uncertainty as to whether these doubtful roots may eventually turn out to be real or not, reference to the criteria should always be made at each completed step. Thus, take the example at page 308 of the "Theory of Equations," namely,

$$12x^3 + 24x^2 - 58x + 25 = 0,$$

in which it is doubtful whether the two roots indicated between ·7 and ·8 are real or imaginary. Proceeding on the supposition that they are real, the first figure of each root must be ·7. Diminishing by this number, the transformed equation is

$$12x^3 + 49{\cdot}2x^2 - 6{\cdot}76x + {\cdot}276 = 0.$$

The next figure, still presuming the roots to be real, is ·06 ; and the next transformed equation is

$$12x^3 + 51{\cdot}36x^2 - \ 7264x + {\cdot}050112 = 0.$$

Taking the first of these three equations, we see at a glance that 58^2 is greater than 24 × 25, and greater even than four times that product, or 24 × 100 ; so that here there is no indication of imaginary roots. Taking the second equation, the conclusion is similar: it is seen at once that $6{\cdot}76^2$ must exceed not only $4{\cdot}92 \times 2{\cdot}76$, but also three times this product. But as respects the third equation, the conclusion is different; ${\cdot}7264^2$ is obviously *less* than $5{\cdot}136 \times {\cdot}50112$; and therefore we may be certain that the two roots, hitherto in doubt, are imaginary.

Suppose, however, instead of the inferior limit ·7, between which and ·8 the roots are indicated, we had taken the superior limit, and had diminished each root by ·8 : the transformed equation would have been

$$12x^3 + 52{\cdot}8x^2 + 3{\cdot}44x + {\cdot}104 = 0,$$

in which the condition of imaginarity is satisfied; for three times $52{\cdot}8 \times {\cdot}104$ is greater than $3{\cdot}44^2$, since it is pretty obvious that $3{\cdot}44^2$ cannot be so great as even 15.

(30) Of all the known methods for determining the numerical values of such of the roots of an equation as may be real—after by trial, or by a previous analysis, the intervals in which they lie are ascertained, Horner's method of continuous development is to be preferred. And if this method be employed in combination with the criteria of imaginarity in the case of doubtful roots, we shall always be led to satisfactory conclusions respecting all the roots of a numerical equation.

But even without applying these tests, the true character of the doubtful roots may always be discovered by proceeding onwards with the development exactly as we should do if the doubtful roots were known to be real. For in this way we shall invariably arrive at an absolute term in a transformed equation which, if the roots be imaginary, will be seen to be irreducible to zero, however far the approximative process be continued; that is, we shall have evidence that the absolute term (the final number) in each successive transformed equation *must* tend, as the work proceeds, not to zero, but to a finite limit, beyond which, towards zero, the absolute term cannot pass—a conclusive indication that the roots in the interval we are thus contracting are imaginary. But the tests of imaginarity will generally enable us to resolve the doubt at an earlier stage of the work.

It is always better, in Horner's process, to develop positive roots only; and, with this object in view, to convert negative roots into positive by changing alternate signs: the passage of a pair of roots will then always be indicated by the *loss* of two variations, and there can never be a *gain* of variations. We speak here of the passage of but a single pair of roots in thus continuously proceeding from the inferior towards the superior limit of the interval; but, in equations of high degree, several pairs may pass simultaneously, and consequently as many pairs of variations be lost. Such will always happen when there are four, six, &c., equal roots, or when either of the functions is made up of equal quadratic factors, whether the roots of these be real or imaginary. [The consideration of these cases of equal roots is postponed to a NOTE at the end.] It is scarcely necessary to remark here that the process for computing the function $f(x)$, for any value of x, by Horner's method, supplies, in its progress, the computation, in order, of the subordinate functions

$$\ldots\ldots,\quad \frac{1}{2\cdot3}f_3(x),\quad \frac{1}{2}f_2(x),\quad f_1(x)\ldots\ [1]$$

for that value of x. If the interval which is doubtful, as respects the equation $f(x) = 0$, is doubtful also in reference to one or more of the inferior equations $f_1(x) = 0$, $\frac{1}{2}f_2(x) = 0$, &c.— a circumstance which the rule of signs of Budan will apprise us of upon comparing the signs due to one limit of the interval with those due to the other limit, then the *first* of the functions [1]—the function $f_3(x) = 0$, suppose,* which, when equated to zero, has two doubtful roots in the same interval as each of the functions following, up to $f(x) = 0$, inclusive, must be such that the immediately preceding function $f_4(x) = 0$, will have one, and only one, root in this same interval.† And it is always to *this* real root that our approximation tends as we work onwards towards

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* For simplicity, we here suppress the fractional multiplier: the roots of an equation being the same whether the significant member of it be multiplied by a number or not, and as it iswith roots only, and not with numerical values of functions, that we are here concerned, the multiplier alluded to may be dismissed.

† See "Theory of Equations," p. 170.

$f(x)$. It is thus that the roots of $f(x) = 0$, lying in the same interval, are separated, when they are real, and when not real, are shown to be imaginary by the continuous tendency of $f(x)$, after a certain stage of the process, not to zero, but to a finite limit; and it is obvious that such tendency there must necessarily be whenever the roots of $f(x) = 0$, in the interval under examination are imaginary, whether the roots of $f_3(x) = 0$, in that interval be imaginary or not. If these latter roots be real, the process will separate them; if imaginary, $f_3(x)$ will itself also tend to a finite limit; and a pair of imaginaries in $f(x) = 0$ will be indicated. In the former case, after the passage of the real root of $f_3(x) = 0$, if a real root of $f(x) = 0$ have not *also* passed, the process is to be renewed, the approximation being now directed to the development of the remaining single root of $f_3(x) = 0$ in the remaining interval; just as at first it was directed to the development of the single root, in the original interval, of $f_4(x) = 0$; until, in the case of the roots of $f_2(x) = 0$ continuing doubtful, notwithstanding this further contraction of the interval, *these*, if real, become separated; and so on, up to $f_1(x) = 0$, and $f(x) = 0$.

In this way, the doubtful roots, if they turn out to be real, are continuously approximated to, however closely they may lie together; and we now proceed to show that the criteria established at the outset of this paper—without even regarding the tendency of the absolute number,*— can never fail to detect their existence whenever the doubtful roots are imaginary:—to show, in fact, that whatever be the character of a pair of imaginary roots of the proposed equation, that pair will always be replaced by, or give rise to, a primary pair in a more or less remote transformed equation; and this, we think, is an important truth.

In order to prove it, however, we must premise, what has been clearly enough proved by Fourier, that in the operation of continuous development, briefly described above, the limits of the doubtful roots become so contracted as we proceed, that not only do those limits exclude all roots except one root, of the function (taking the series of functions from right to left) immediately beyond the last of [1], into which the doubt enters, but they also exclude every root of the immediately next following function: in other words, the interval becomes at length so contracted that in the passage over it, while two variations are lost in the series of signs under the functions [1], reckoning onwards from left to right, up to the above-mentioned doubtful function inclusive, only one variation is lost in the series terminating at the immediately antecedent function: and no variation at all is lost in the series ending at the function immediately before this.†

(31). Now let $f_m(x)$ be the function described above as the last of the consecutive functions [1] in which doubt enters; and let us assume that the two roots of $f_m(x) = 0$, in the interval under examination, are imaginary: then the operation of continuous development, as explained above, will conduct us eventually to a transformed equation, such that the three coefficients under the functions,

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* Nevertheless, it is always advisable to take note of this tendency as the approximation proceeds. The fluctuations of the absolute number, in its passage from one transformation to another, may give early indication of the true character of the doubtful roots, although it be not to a root of $f_1(x) = 0$ that the approximation is directed.

† See "Theory of Equations," p. 172, *et seq.*

$$f_{m+2}(x), \quad f_{m+1}(x), \quad f_m(x)$$

will satisfy the condition of imaginarity; that is, the triad will indicate the presence of a primary pair of imaginary roots in the transformed equation, as may be proved as follows:

The approximation being to the real root (r) of $f_{m+1}(x) = 0$, lying in the contracting interval, the coefficient under $f_{m+1}(x)$ will continue tending to zero, whilst the coefficient under $f_m(x)$ is approaching a finite limit. Moreover, in the passage over the interval $[r-\delta, r+\delta]$, δ may become so small that, in the terms of the transformed equation, up to the term under $f_{m+2}(x)$ inclusive, no variation shall be lost: but, taking in the two terms next following, two variations are lost in the passage over the root r, since $f_{m+1}(x)$ changes sign in this passage, whilst the signs of the preceding functions, as well as the sign of $f_m(x)$, remain unchanged. Now this cannot possibly be unless for the transformation $r - \delta$ the collocation of signs under the three functions is either $+ - +$, or $- + -$; for otherwise there could not be two variations to lose. Hence, in passing over the interval, $[r - \delta, r + \delta]$, the signs of the first and third of the coefficients under the above three functions continue to be like signs; and as the middle coefficient vanishes in this interval, it follows that not only at, but before and after this evanescence, the square of the middle one of the three coefficients must be *less* than the product of the other two coefficients. The triad must therefore satisfy the condition of imaginarity; and must do so all the earlier in the process of continuous development, inasmuch as the square has for multiplier a number *less* than the multiplier for the product by the number $n + 1$ (formulæ 5, p. 344). So soon as the triad of coefficients satisfies this condition of inequality, a stop may be put to the work, provided but one pair of doubtful roots lies in the interval. We should know that, in the remaining portion of the interval, a pair of imaginary roots would exist for each succeeding function. If, however, there are other pairs of roots in the interval under examination, the transformation must be completed, and the development be proceeded with, in order to ascertain, from the variations lost between $r - \delta$ and $r + \delta$, whether *additional* pairs of imaginary roots are also indicated by the passage of the root r.

Whatever pairs, besides these additional pairs (if any), may still be indicated in the original interval $[a, b]$, they are to be sought in the partial interval $[r, b]$, by proceeding in the same way as at first. We have only further to observe that, when the *leading* triad of the proposed equation satisfies the condition at page 343, we know that the leading triad for *every* transformation will also satisfy it (27). But the pair of imaginary roots in $f(x) = 0$, which this triad indicates, lies in the interval

$[a', b']$, which, embracing two roots of $f(x) = 0$, embraces also the root of the middle equation of the first degree.* It thus appears that those imaginary roots of an equation which have not, at first, the character of what we have called primary pairs, become convertible into primary pairs by the same process of continuous approximation by which they would be separated and computed if they were real. And we submit that all desirable extension and efficiency is thus given to Horner's method of development, and to the general criteria at page 344.

(32) As to the practical operation of carrying on the development adverted to, when two or more roots are indicated in the same interval, and are long in separating when real, or in disclosing their character when imaginary, we must refer for the necessary directions—more especially as to the trial divisors for facilitating the discovery of the successive figures of the real root actually approximated to, to "The Theory and Solution of Equations of the Higher Orders," pp. 259-263. But we may add here that, in testing a triad of coefficients by the condition of imaginarity, if the figures of these coefficients are numerous, and it be seen necessary to compute with some degree of precision, the squaring and multiplying may be tedious operations. In such cases we would recommend a shorter method of proceeding, thus:—Let the product, with its proper Newtonian multiplier, be represented by pPP', and the square, with its proper multiplier, by qQ^2: then the condition

$$pPP' > qQ^2 \text{ implies } \frac{pP}{Q} > \frac{qQ}{P'},$$

and these division operations being carried on, a figure at a time alternately, we shall find which quotient exceeds the other without computing even a single superfluous figure.

(33) We shall terminate this paper with the investigation of a general formula for determining the character of the roots of a complete cubic equation, independently of actual development.

If either of the two triads of a cubic equation satisfy the condition of imaginarity, no special formula for this purpose will be necessary: we have therefore only to provide for the case in which both triads fail, or in which the square of the middle term of each (with its proper factor), minus the product of the extremes (with its proper factor), is a positive quantity. Let

$$\begin{aligned} P &= A_3x^3 + A_2x^2 + A_1x + A_0; && \text{or, page 367,} \quad X_0 \\ Q &= 3A_3x^2 + 2A_2x + A_1 && \quad ,, \qquad X_1 \\ P' &= 3A_3x + A_2 && \quad ,, \qquad X_2 \end{aligned}$$

$$\because\ Q^2 - 3PP' = (A_2^2 - 3A_1A_3)x^2 + (A_1A_2 - 9A_0A_3)x + (A_1^2 - 3A_0A_2) \,..\, [1]$$

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* The passage of this root being attended with the loss of two variations; that is, two variations are lost in passing from $r - \delta$ to $r + \delta$.

Now, this expression is positive for every real value of x, provided it be either a complete square, or that it satisfy the condition

$$4(A_2^2 - 3A_1A_3)(A_1^2 - 3A_0A_2) > (A_1A_2 - 9A_0A_3)^2 \ldots \qquad [2]$$

and it cannot be positive, for *every* real value of x, unless one or other of these conditions hold. Hence, when all the roots of $P = 0$ are real, [1] must be a complete square, or else the condition [2] must have place, and conversely; so that the condition which must be satisfied when a pair of roots is imaginary, and which cannot be satisfied unless there be an imaginary pair, is

$$4(A_2^2 - 3A_1A_3)(A_1^2 - 3A_0A_2) < (A_1A_2 - 9A_0A_3)^2 \ldots \qquad [3]$$

The criterion of imaginary roots for an incomplete cubic, is, of course, but a particular case of this more general condition; the case, namely, in which $A_3 = 1$, and $A_2 = 0$; for making these substitutions, [3] becomes

$$-12A_1^3 < (9A_0)^2, \quad \text{or} \quad \left(-\frac{A_1}{3}\right)^3 < \left(\frac{A_0}{2}\right)^2$$

as at p. 351; A_1, A_0, here, being p, q, there.

It follows from the above conclusions that when the roots of $Q^2 - 3PP' = 0$ are imaginary, those of $P = 0$ must all be real; and that when the roots of $Q^2 - 3PP' = 0$ are real, two roots of $P = 0$ must be imaginary, and *vice versâ*, unless the roots of $Q^2 - 3PP' = 0$ are equal roots; and equal roots they will always be whenever $P = 0$ has two equal roots, the equal pair in the former of these equations being the same as the equal pair in the latter. For one of the two equal roots of $P = 0$, namely, $x = r$, must enter the equation $Q = 0$; so that Q^2 and $3PP'$ are each divisible by $(x - r)^2$; therefore r, r, are the two roots of the quadratic $Q^2 - 3PP' = 0$. In the case supposed, therefore, [1] is always a complete square:* hence, if the equation $P = 0$ have equal roots r, r,

$$A_2^2 - 3A_1A_3, \text{ and } A_1^2 - 3A_0A_2$$

must be squares; and

$$r = \sqrt{\frac{A_1^2 - 3A_0A_2}{A_2^2 - 3A_1A_3}}$$

and whether the sign of r is to be positive or negative will be at once ascertained from the signs in the proposed equation, by the rule of Descartes, as all the roots must be real.

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* Of the form $m^2(x - r)^2$.

If $A_2 = 0$, then $-3A_1A_3$ is a square, in the case of equal roots, and

$$r = \sqrt{-\frac{A_1}{3A_2}} = \frac{1}{3}\sqrt{-\frac{3A_1}{A_2}}$$

If all the roots of $P = 0$ are equal, each coefficient of [1] will be zero; $Q^2 - 3PP'$ being then identically zero.

NOTE 1.

In the foregoing discussion, we have not specially considered the cases in which equal roots enter an equation, with the exception of what has just been said as to the cubic. Such special consideration of these cases, when actual solution of the equation is the sole object in view, we do not regard as at all necessary; and we cannot but think that a great deal of labour is sometimes expended, with but little profit, in trying to find out, by tedious common measure operations, whether an equation has roots strictly equal or not.

If equal roots really enter an equation, the approximation to that one of them which always enters singly into an antecedent derived equation, must cause, not only the results in the corresponding column of work to approximate to zero, but also the results in each of the subsequent columns, up to the final column, or that which computes $f(x)$. The simultaneous tendency to zero of the results under $f(x)$, $f_1(x)$, $f_2(x)$, &c., always of course indicates so many roots either accurately equal, or nearly equal; unless, indeed, the tendency to zero in $f(x)$ should cease, after a certain number of steps, and thus conduct us to the condition of imaginary roots.

When, however, the approximation to the single real root here alluded to has been carried on so far that the incomplete development would be regarded as a value sufficiently near to the complete root of $f(x) = 0$, if this were the only root in the interval; then, although the approximate root neither separates the other roots of $f(x) = 0$, nor yet conducts to the condition of imaginary roots, we may, nevertheless, discontinue the development, and may safely regard the value obtained as a close approximation to one, or two, or three, as the case may be, of the values of x which satisfy the equation $f(x) = 0$.

For the roots in question, having been developed up to whatever number of figures may have been settled upon at the outset, as sufficient for the purpose in hand, what can it matter whether the superfluous figures which follow those already found to be common to the two or more roots, are the same, or different for those roots? The roots are practically equal if the figures which completely express them are of no practical value beyond those thus far found to coalesce, or to be common to them all; whether the more remote figures agree or disagree can be of no moment in reference to the object in view; since, agree or not, they are confessedly useless.

We submit that there is no difference of opinion as to Horner's being the best method of computing the real roots of a numerical equation of an advanced degree by continuous approximation; and with approximations only, in all those instances where a root has interminable decimals, we must be content, even though those interminable decimals may be but a very simple vulgar fraction in another form. As is usually the case with general methods of computation, in whatever department of practical mathematics they are proposed, there will always be particular examples that might be better treated by particular rules. The present writer is not likely to be charged with undervaluing Horner's method: he believes that its merits are such that, as a *general* method, it will never be superseded. But, however high one's estimate of any practical process may be, it is right fairly to state its inconveniences in particular cases, as well as its general advantages; and an inconvenience in Horner's process, it certainly is—we think the only inconvenience—that a fractional root has to be developed in decimals.

Suppose, for example, one of the roots of an equation to be $\frac{1}{7}$: this root, by Horner's method, would be determined in the approximate form ·142857 . . . . , and if the equation were of an advanced degree, a good deal of numerical work would be required to obtain this approximative value of $\frac{1}{7}$. If the development were to be extended two or three places further, the recurrence of the figures would, no doubt, suggest the equivalent fraction; but fractions may readily be assigned the equivalent decimal of each of which would not be seen to be a recurring decimal till many more figures were computed. However, if it be of no practical consequence in the inquiry before us that a root with interminable decimals, and which is *not* the development of any finite fraction, should be approximated to beyond, say, six places of decimals, neither can it be of any practical consequence that ·142857 should replace $\frac{1}{7}$ in that inquiry.

Viewing the matter generally, in reference to incommensurable roots, it would be more strictly accurate to regard our approximations—not as approximations to the exact roots of the equation we are dealing with (for it may not have exact roots—roots expressed in finite numbers), but to consider each as the complete or exact value of a root of an approximate equation—this approximate equation differing from the equation proposed only in its final term or absolute number. The amount of the difference may be made smaller than any assignable decimal; for the development of the incommensurable root may be carried to such an extent, that the final term of the transformed equation, at which the operation is stopped, may differ from zero by as small a quantity as we please; and the root thus far developed will be a complete root of that approximate equation which would arise from merely correcting the absolute term of the proposed equation by the small decimal alluded to. What is here said as to a single incommensurable root applies, of course, equally to two or more roots which agree in their leading figures to the extent mentioned. These are equal roots of an approximate equation, and have the same claim to be considered equal roots of the proposed equation as either of them has to be considered *a* root of it. The conclusion is the same, whether at a more remote figure the roots would separate, or the condition of imaginary roots, hitherto delayed, be afterwards fulfilled *for a final triad.* In either case a real value is found which satisfies an equation so nearly coincident with the equation proposed that, since approximations only are attainable, the two equations may be regarded as identical—in so far, at least, as the roots thus far common to both are concerned.

No doubt, in thus prosecuting the development of one of a pair of contiguous roots, there may be abiding uncertainty as to whether the roots are strictly equal or not; for the decimals being interminable, all that we can affirm, however far these decimals are carried, is, that thus far, at least, the roots are undistinguishable from equal roots—supposing, that is, that a separation has not yet taken place. But in the case of a pair of imaginary roots, there need never be *abiding* uncertainty at all. The approximative process may fail to separate a pair of real roots—for they may not be separable; but a pair of imaginary roots must, sooner or later, unfold their character as such, by means of the tests of imaginarity. There are no such tests for incommensurable equal roots: by continuous approximation, $f(x)$ may, in the one case, continuously tend to zero *interminably;* in the other case, it cannot approach zero within a certain finite limit;* but, in either case, the process may be stopped when a sufficient number of decimals is obtained; and the inexact root, thus far developed, will be an exact root of an equation so nearly coincident with that proposed, that it may be substituted for it without appreciable error, in so far as concerns the particular roots in question.

It should be observed, however, that the approximate equation, of which a partially developed root of the proposed equation is an exact root, is not precisely the same as the approximate equation for another partially developed root; since for different inexact roots the correction for the absolute number of the proposed equation will most likely be different; and similarly as respects different groups of approximate equal roots. But unless the correction in each case be so small as to be of no practical consequence, the approximate equation will not have approached near enough to the proposed to justify its being substituted for it. All that the present Note affirms is, that a pair of real roots *may* have so many leading figures in common, or a pair of imaginary roots *may* have the imaginary element so insignificant, that both in the one case and in the other the roots may be regarded as real and equal—satisfying the conditions of the equation with as much precision as the received approximate value of any single incommensurable root of it satisfies those conditions. And we therefore think that the indiscriminate rejection of roots involving an imaginary element, regardless of the influence of this element on the coefficients of the equation, is unjustifiable, and, moreover, inconsistent, where approximate values only of the real roots—that is to say, exact values of the roots of only approximate equations—are received.

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* If it be not the final triad, but a preceding triad, which indicates the imaginarity, then, as the approximation will not be directed to a root of $f_1(x) = 0$, the successive values of $f(x)$ may even diverge from zero, though not beyond a finite limit.

Some remarks on this subject will be found in Peacock's paper, in the "Report of the Third Meeting of the British Association," p. 349.

There is, however, one case of equal roots which deserves special notice—namely, the case in which the roots are all equal imaginary pairs.

Let the equation be one of the fourth degree, made up of two equal quadratic factors (X) each having a pair of imaginary roots: the equation is then $XX = 0$. Taking the successive limiting equations (or differentiating), and remembering that X being of the second degree, we must have, at the third step of the operation, $X''' = 0$, the limiting equations will be

$$2XX' = 0 \quad \dots\dots [1]$$
$$2X'X' + 2XX'' = 0 \dots\dots [2]$$
$$4X'X'' + 2X'X'' = 0,$$

that is, $$6X'X'' = 0 \quad \dots\dots [3]$$
$$6X''X'' = \text{a positive number.}$$

Now, the root of the equation $X' = 0$, of the first degree, is the real root of [1] of the third degree; and it is equally the root of [3]; which is also of the first degree, X'' being a constant number. Hence, if the roots $XX = 0$ be diminished by the root of the simple equation $X' = 0$, that is, if we cause the second term of the proposed equation to disappear, the fourth term will vanish also; they will vanish, moreover, between *plus* signs; for X^2 is always plus, and in [2] X'^2 and X are both always plus; and X'' is a positive number. When, therefore, the imaginaries are equal pairs, the alternate terms, in the transformation which removes the second term, are zeros, each zero being between like signs, and we know that whenever this happens the roots are all imaginary. The same has place when the equation is $X^2 + N = 0$, N being any positive number, though the pairs are not then equal pairs.

The conclusion is the same, whatever be the number of equal quadratic factors $XXX \dots$ of the above kind; that is, if the roots of the equation be each diminished by the root of the simple equation $X' = 0$, the last of the derived equations, the result will be an equation in which the alternate terms will be zeros, each zero being between like signs. For, let the number of equal quadratic factors be n; then, using the notation above, no X can appear in any of the derived equations with more than two dashes, because X''' is zero. In the first derived equation, there occurs but one dash in each term; in the second, there are two dashes in each term; in the third, three; and so on up to the final step, in which there are $2n$ dashes; this last result being a positive number: moreover, each term consists of a group of n X's.

Now, it is plain, when in each of the terms, or individual groups, entering a derived equation, the number of dashes is odd, that X' must, of necessity, enter that group once, or some odd number of times; and that when the dashes in each group of X's, in a derived equation, are even in number—since that even number can be made up without any X at all—there must be one group from which X' is absent. It follows, therefore, that the root of the simple equation $X' = 0$ is equally a root of every ascending equation of an odd degree; but that for this, and for all real values of x, every one of the functions of even degree is positive; hence, if we diminish each of the roots of the proposed equation by the root of $X' = 0$, we shall arrive at a transformed equation in which the alternate terms will be zeros, each zero being between plus signs.

In diminishing by the root of $X' = 0$, should it consist of two or more figures, Newton's Rule, in the case here considered, is of special service; as the conditions of imaginarity are likely to hold and fail alternately before the full development of the root of $X' = 0$ is completed. And the same may be said in reference to equations made up of quadratic factors furnishing imaginary pairs not strictly identical, but only nearly so. But in every application of this rule to the determination of the number of imaginary pairs in an assigned interval $[a, b]$, care must be taken that pairs outside that interval are not included in the enumeration; that is, that the evanescencies—which Newton's Rule anticipates—are not any of them delayed *beyond* the limit b; such as are so delayed imply pairs belonging, of course, to the succeeding interval $[b, c]$.

NOTE 2.

It may be well, before closing this paper, to give a short practical illustration of one or two of the theoretical principles established in the latter articles of it. The equation of the fourth degree, at p. 368, is well adapted to this purpose, since the roots are all doubtful, and all lie within the narrow limits $[0, 1]$.

The inquiry is—Does an imaginary pair enter this equation? and if so, by which triad of terms is the entrance of the pair first indicated? It cannot be by the leading triad, from the principle at (27); that is, an imaginary pair cannot enter the derived equation $X_2 = 0$. Can a pair enter the derived equation of next higher degree—namely, $X_1 = 0$?

The roots of the derived quadratic $X_2 = 0$—that is, of $8x^2 - 10x + 3 = 0$—are readily found to be $\cdot 5$, and $\cdot 75$. Taking the smaller of these as a transforming number, and working up to X_0, if imaginarity is *not* indicated in $X_1 = 0$, real roots of $X_0 = 0$ (if such exist) may separate.

$$\begin{array}{rrrrl}
4 & -10 & +9 & -3 & +\cdot 25(\cdot 5 \\
 & 2 & -4 & 2\cdot 5 & -\cdot 25 \\
\cline{2-5}
 & -8 & 5 & -\cdot 5 & \quad 0 \\
 & 2 & -3 & 1\cdot & \\
\cline{2-4}
 & -6 & 2 & \cdot 5 & \\
 & 2 & -2 & & \\
\cline{2-3}
 & -4 & 0 & & \\
 & 2 & & & \\
\cline{2-2}
 & -2 & & &
\end{array}$$

Here we see that $\cdot 5$ is a root, not only of $X_2 = 0$, but also of $X_0 = 0$; and that there is no indication of imaginarity in $X_1 = 0$. A real root of this equation is, however, passed over; and it further appears that, for the transformation $\cdot 5 - \delta$, one variation, in the entire series of signs, would be lost—the signs for this transformation being $+ - + + -$, as is obvious. Hence, one real root of the proposed equation $X_0 = 0$, lies between 0 and $\cdot 5$; a second real root, as just seen, being $\cdot 5$ itself.

Again, diminish the roots by $\cdot 7$.

$$\begin{array}{rrrl}
4 - 10 & +9 & -3 & +\cdot 25 \ (\cdot 7 \\
2\cdot 8 & -5\cdot 04 & 2\cdot 772 & -\cdot 1596 \\
\hline
-7\cdot 2 & 3\cdot 96 & -\cdot 228 & \cdot 0904 \\
2\cdot 8 & -3\cdot 08 & \cdot 616 & \\
\cline{1-3}
-4\cdot 4 & \cdot 88 & \cdot 388 & \\
2\cdot 8 & -1\cdot 12 & & \\
\cline{1-2}
-1\cdot 6 & -\cdot 24 & & \\
2\cdot 8 & & & \\
\cline{1-1}
1\cdot 2 & & &
\end{array}$$

As before, one root, and one root only, of $X_1 = 0$, is overstepped; but there is indication sufficient that the other two roots of this equation—and, therefore, two roots of $X_0 = 0$—are imaginary. Hence, the equation has two real roots, $\cdot 5$ and $[0, \cdot 5]$, and two imaginary roots. The approximate value of the latter of these two real roots, found in the usual way, is $\cdot 12256$: it is somewhat more expeditiously found by employing for the purpose the depressed equation of the third degree, $4x^3 - 8x^2 + 5x - \cdot 5 = 0$,* as given by the first row of coefficients in the former of the two operations above.

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* We need scarcely remind the reader that the first member of this equation is the quotient arising from dividing the first member of the given biquadratic equation by $x - \cdot 5$.

In the foregoing operations, we have analyzed the interval [0, 1], within which the four roots are all comprised, by commencing with the roots of the quadratic $X_2 = 0$, seeing that these roots are real, and their values so easily determined. There is, of course, no necessity to reach this quadratic through the descending steps at p. 368; we may *ascend* to it by means of the three leading coefficients $4 - 10 + 9$; thus the simple equation is $4 \times 4x - 10 = 0$, that is, $16x - 10 = 0$; and the quadratic, $\frac{3 \times 16}{2}x^2 + 3 \times 10x + 9 = 0$, that is, $24x^2 - 30x + 9 = 0$. But if, without regarding this quadratic, we had commenced with the leading figure of the root of the *simple* equation, namely, $\frac{10}{4 \times 4} = \cdot 6 \ldots$, a proceeding which would have been in strict accordance with the general directions at p. 370, the corresponding transformed equation would have been

$$4x^4 - \cdot 4x^3 - \cdot 36x^2 + \cdot 454x + \cdot 0484 = 0\,;$$

so that the original interval would then have been subdivided into the two partial intervals [0, ·6] and [·6, 1], each comprising two roots; and the character of each pair would become known by contracting each of these intervals as above.

NOTE 3.

It is here proposed to prove that whenever the condition of imaginary roots holds or fails for any triad of the functions X_p, X_{p+1}, X_{p+2}, &c., as deduced from the primitive X_0, for an assigned value of x, it will in like manner hold or fail, for the same value of x, for the corresponding triad (the first, second, third, &c.) when X_p is taken for the primitive function.

If we take X_1 for the new primitive, the series of expressions furnished by X_1 and its derivees will be the original series X_1, X_2, X_3, &c., multiplied respectively by 1, 2, 3, 4, 5, &c., taken in order.

If X_2 be taken for the new primitive, X_2 and its derivees will be the original X_2 X_3, X_4, &c., multiplied, in order, by 1, 3, 6, 10, 15, &c.

And if X_3 be taken for the new primitive, X_3 and its derivees will be obtained by multiplying the original X_3, X_4, X_5, &c., by 1, 4, 10, 20, 35, &c., taken in order, and so on.

This will be readily seen to be the case from a mere inspection of the several developments.

Now the general expression for the m^{th} term in any one of these series of figurate numbers is

$$\frac{m(m+1)(m+2)(m+3)\ldots(m+p-1)}{1.\ 2.\ 3.\ 4\ldots p} \ldots [A],$$

p being of permanent value for the same series, whatever be m, or the number denoting the place in the numerical scale, of any single term $[A]$ in that series; thus, for the first series, $p = 1$; for the second, $p = 2$; for the third, $p = 3$; and so on.

If we take the two terms B, C, next following this m^{th} term $[A]$, the first of these, B, will evidently be the expression $[A]$ with the leading factor, m, suppressed, and the new factor $(m + p)$ annexed; and the next term, C, will be what $[A]$ becomes when the two factors $m(m + 1)$ are removed, and the two, $(m + p)(m + p + 1)$, introduced.

Hence we have the conditions

$$\frac{(m+1)C}{B} = m + p + 1, \quad \frac{mB}{A} = m + p \quad . \; . \; . \; [1]$$

$$\therefore \{(m+1)\,AC,\; mB^2\} = \{m + p + 1, m + p\}\; AB \; . \; . \; . \; [2],$$

m being the numerical place of A in the series, and p denoting the place, or order of the series itself.

[We may here notice, in passing, as an inference from the relation [1], that the place (p) of any one of the series being given, we can readily write the entire series from the beginning, or can extend it, when leading terms of it are already written, since $B = \frac{(m+p)A}{m}$; and p, it will be observed, is always equal to the second term *minus* 1: thus, for the third series the first two terms are 1, 4; the next term is $\frac{(2 + 3)4}{2} = 10$; the next $\frac{(3 + 3)10}{3} = 20$; and so on.]

From the numerical equivalence [2], the proposition enunciated above may be deduced as follows:—

Let X_p (taken as primitive), and its derivees, be denoted by $(X_p)_0$, $(X_p)_1$, $(X_p)_2$, &c.: then calling the highest exponent of x in X_p, n', we shall have the several conditions (p. 367) for these new functions, by substituting them for the functions, X_0, X_1, X_2, &c., in the formulæ referred to, provided we put n' for n throughout. But since, by the property adverted to at the outset $(X_p)_0$, $(X_p)_1$, &c., are no other than X_p, X_{p+1}, &c., multiplied, in order, by the figurative numbers, 1, $1 + p$, &c., if A be the m^{th} number in the series, B, C, being the two numbers immediately following, then, as $(X_p)_{m-1}$ is the m^{th} function in the series $(X_p)_0$, $(X_p)_1$, $(X_p)_2$, &c., we shall have

$$ACX_{p+m-1}X_{p+m+1} = (X_p)_{m-1}\,(X_p)_{m+1}, \text{ and } B^2X^2_{p+m} = (X_p)^2_m.$$

Now, by the formula [4] at p. 344, the condition of imaginarity for the three functions here last written is

$$(m + 1)(n' - \overline{m - 1})(X_p)_{m-1}(X_p)_{m+1} > m(n' - m)(X_p)^2_m \; . \; . \; . \qquad [3]$$

and the condition of imaginarity for the three functions of which these are the multiples (A, B, C), is the condition within the brackets in the following expression, namely,

$$\{(p+m+1)(n-p-\overline{m-1})X_{p+m-1}X_{p+m+1} > (p+m)(n-p-m)X^2_{p+m}\}AB \; [4]$$

and by the property [2] these two expressions [3], [4], are equal; for, puttig in the former $n-p$ for n', which it is, we see that the two members of [3] and the two of [4] are respectively what the following, namely,

$$(m+1)AC,\ mB^2,$$
$$\{m+p+1,\ m+p\}AB$$

become, when the first member of each is multiplied by the same number, $(n-p)-(m-1)$; and the second member of each by the same number, $n-p-m$. Hence if the condition of imaginarity hold or fail, for any value of x, for the three functions in [3], the condition will, in like manner, hold or fail for the same value of x, for the three functions of the same degree in [4], and, moreover, the two members of the condition [3] are the two members of the condition [4], each multiplied by AB.

As a practical illustration of this, in a particular case, let X_3 be the function taken as primitive; then $p=3$, and $n'=n-3$; also, for this value of p, the figurate multipliers are 1, 4, 10, 20, 35, &c.

Taking the first triad of these for A, B, C; then the second triad; and so on, we have

1st. $\{2(n-3)10X_3X_5 > (n-4)16X^2_4\} =$
$\{2(n-3)(X_3)_0(X_3)_2 > (n-4)(X_3)_1^2\} =$
$\{5(n-3)X_3X_5 > 4(n-4)X_4^2\}4.$

2nd. $\{3(n-4)80X_4X_6 > 2(n-5)100X_5^2\} =$
$\{3(n-4)(X_3)_1(X_3)_3 > 2(n-5)(X_3)_2^2\} =$
$\{6(n-4)X_4X_6 > 5(n-5)X_5^2\}40.$

3rd. $\{4(n-5)350X_5X_7 > 3(n-6)400X_6^2\} =$
$\{4(n-5)(X_3)_2(X_3)_4 > 3(n-6)(X_3)_3^2\} =$
$\{7(n-5)X_5X_7 > 6(n-6)X_6^2\}200.$

and so on, conformably to the general conclusion above, namely, that [3] = [4].

The object of the foregoing investigation is to prove that when a limiting equation $X_p=0$, derived in the ordinary way from the primitive equation $X_0=0$, has imaginary roots indicated between assigned limits, in contracting these limits by Horner's process—whether we operate upon the equation $X_0=0$ itself, or upon the derived equation of inferior degree, $X_p=0$, the indications of imaginary roots will present themselves, in both operations, at precisely the same step of the two processes. Since the numbers, resulting from a transformation in the one operation are all different (except those under X_p, and such as may be zero from the numbers resulting from the corresponding transformation in the other operation, there might be uncertainty, in the absence of proof, as to whether the indications of imaginarity, when we operate upon $X_0 = 0$, might not be delayed beyond the step at which they would offer themselves if we were to operate upon $X_p = 0$ itself; we now see that such delay can never occur; but that the fulfilment or the failure of the prescribed condition, for any triad of coefficients in one of the transformations of $X_0 = 0$, implies the like fulfilment or failure for the corresponding triad of coefficients, in the corresponding transformation (by the same number) of $X_p = 0$.

And hence the remarks, at p. 378, respecting equal and nearly equal imaginary pairs *entering* $X_0 = 0$, equally apply whenever such pairs enter a derived equation.

XXXIX.—On an Ogham-inscribed Monument in Glen Fais, County Kerry. By Richard Rolt Brash, M. R. I. A.

[Read November 9, 1868.]

On November 8th, 1858, a paper of considerable interest was read before the Royal Irish Academy by the late Venerable the Archdeacon of Ardfert, Dr. Rowan, giving an account of the discovery by that gentleman of a remarkable inscribed monument in Glen Fais, and of the historic locality in which it was found. As the readings given in the Archdeacon's paper appeared to me unsatisfactory, as also those given in other publications, I was anxious to obtain a personal inspection of the stone in question, to ascertain if the published copies, as well as others in my possession, were correct, as I have had abundant reason to distrust copies of Ogham inscriptions, unless made by very experienced and trustworthy Oghamists. Being on an antiquarian tour in the barony of Corcaguiney, in July of the present year, I had an opportunity of gratifying my desire, by visiting the locality of the monument, which I found lying prostrate in a grass field in the townland of Camp, a portion of Glen Fais, or, as it is locally pronounced, Glenaish, under the west face of Caher Conrigh mountain. It lies about twenty yards inside the fence, to the left of the public road winding up the glen, and about ten minutes' walk from Camp Post-office; distant from Tralee nine miles. The locality will be found on sheet No. 37 of the Ordnance Survey of Kerry, on which, however, the monument is not marked. It is an irregular flag-shaped monolith, measuring in length eleven feet five inches, and in extreme breadth five feet nine inches, and varying in thickness from ten to eighteen inches; it is a hard, compact, close-grained red sandstone, the inscription being on an obtuse angle on the face of the stone towards the left, and about midway in the length of the monolith. The engraving which accompanied Dr. Rowan's paper ("Proc. R. I. A.," vol. vii. p. 104), is a fair representation of the stone, while the inscription is, I am happy to say, accurately copied. The line on which the characters run is